

STABLE FLATNESS OF NONARCHIMEDEAN HYPERENVELOPING ALGEBRAS

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ABSTRACT. Let L be a p -adic local field and \mathfrak{g} a finite dimensional Lie algebra over L . We show that its hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$ is a stably flat completion of its universal enveloping algebra. As a consequence the relative cohomology for the locally convex algebra $\mathfrak{F}(\mathfrak{g})$ coincides with the underlying Lie algebra cohomology.

1. INTRODUCTION

Let p be a prime number, let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ be a chain of intermediate complete fields where L/\mathbb{Q}_p is finite and K discretely valued. Let G be a locally L -analytic group.

Cohomology theory for locally analytic G -representations (in the sense of [2]) follows J.L. Taylor's general approach of a homology theory for topological algebras ([9],[10]). With the algebra $D(G, K)$ of K -valued locally L -analytic distributions on G as base algebra the relative homological algebra is developed over the category \mathcal{M}_G of all complete Hausdorff locally convex K -vector spaces admitting a separately continuous $D(G, K)$ -module structure (with morphisms all continuous module maps). Another distinguished locally convex algebra appearing in locally analytic representation theory is the hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$ of $\mathfrak{g} = \text{Lie}(G)$. It equals the closure of the enveloping algebra $U(\mathfrak{g})$ inside $D(G, K)$ and inherits therefore a nontrivial structure as topological algebra. The aim of this brief note is to show that the relative homological algebra of [2] applied to $\mathfrak{F}(\mathfrak{g})$ instead of $D(G, K)$ yields as cohomology groups none other than the usual abstract Lie algebra cohomology of \mathfrak{g} .

We obtain this result via proving that the natural map $\theta : U(\mathfrak{g}) \rightarrow \mathfrak{F}(\mathfrak{g})$ is stably flat (or an absolute localization, cf. [5],[10]). Roughly speaking, a topological algebra morphism $A \rightarrow B$ is stably flat if it identifies the category of topological B -modules with a full subcategory of topological A -modules in a way that leaves homological relations invariant.

We remark that the corresponding result on θ for complex Lie algebras is due to Pirkovskii (cf. [5], Thm. 8.6). We also emphasize that stably flat morphisms between topological algebras are of central importance in complex non-commutative operator theory since they leave the joint spectrum invariant (cf. [5],[10]).

We finally point out that, in order to reduce technicalities, we work over the p -adic field L instead of an arbitrary completely valued nonarchimedean field. All results extend to this more general setting without conceptual differences.

2. THE RESULT

Throughout this note we freely use basic notions of nonarchimedean functional analysis as presented in [4]. We begin by recalling the necessary relative homological algebra following [2],[5],[9]. We emphasize that as in [2],[10] (but in contrast to [5]) our preferred choice of topological tensor product is the completed inductive topological tensor product $\hat{\otimes}_L$. By a *topological algebra* A we mean a complete Hausdorff locally convex L -space together with a separately continuous multiplication. For a topological algebra A we denote by \mathcal{M}_A the category of complete Hausdorff locally convex L -spaces with a separately continuous (left resp. right) A -module structure. Morphisms are continuous module maps and the Hom-functor is denoted by $\mathcal{L}_A(.,.)$. A morphism is called *strong* if it is strict with closed image and if both its kernel and its image admit complements by closed L -subspaces. The category \mathcal{M}_A is endowed with a structure of exact category by declaring a sequence to be *s-exact* if it is exact as a sequence of abstract vector spaces and all occurring maps are strong. Finally, a module $P \in \mathcal{M}_A$ is called *s-projective* if the functor $\mathcal{L}_A(P, .)$ transforms short *s-exact* sequences into exact sequences of abstract L -vector spaces. A *projective resolution* of $M \in \mathcal{M}_A$ is an acyclic complex $P_\bullet \rightarrow M$ where each P_n is *s-projective* and all maps are strong. A standard argument shows that \mathcal{M}_A has enough projectives and that any object admits a projective resolution. As usual for a left resp. right A -module N and M we denote by $M \hat{\otimes}_A N$ the quotient of $M \hat{\otimes}_L N$ by the closure of the subspace generated by elements of the form $ma \otimes n - m \otimes an, a \in A, m \in M, n \in N$. Given a projective resolution $P_\bullet \rightarrow M$ we define as usual

$$\mathcal{T}or_*^A(M, N) := h_*(P_\bullet \hat{\otimes}_A N), \quad \mathcal{E}xt_A^*(M, N) := h^*(\mathcal{L}_A(P_\bullet, N))$$

for $M, N \in \mathcal{M}_A$. These L -vector spaces do not depend on P_\bullet and have the usual functorial properties.

Given a topological algebra A we may form the enveloping algebra $A^e := A \hat{\otimes}_L A^{op}$ as a topological algebra. Given a morphism of topological algebras $\theta : A \rightarrow B$ we may define a functor $B^e \hat{\otimes}_{A^e}$ from the category of A -bimodules \mathcal{M}_{A^e} to the category of B -bimodules \mathcal{M}_{B^e} . The map θ is called *stably flat* (or an *absolute localization*, cf. [5],[10]) if the above functor carries every projective resolution of A^e into a projective resolution of B^e .

As in the introduction we fix a finite dimensional Lie algebra \mathfrak{g} over L and let $U(\mathfrak{g})$ be its enveloping algebra. Denote by $M_{\mathfrak{g}}$ the category of all (abstract) left \mathfrak{g} -modules. Let G be a Lie group over L with Lie algebra \mathfrak{g} . Denote by $D(G, L)$ the algebra of locally analytic L -valued distributions on G (cf. [8]) and by $\mathfrak{F}(\mathfrak{g})$ the closure of $U(\mathfrak{g})$ in $D(G, L)$. According to [1], Prop.1.2.8 $\mathfrak{F}(\mathfrak{g})$ equals the strong dual of $C_1^{an}(G, L)$, the stalk at $1 \in G$ of germs of L -valued locally L -valued functions on G and is therefore (cf. [5],[6]) called the *hyperenveloping algebra* of \mathfrak{g} . According to [1], Thm. 1.4.2 the embedding $\mathfrak{F}(\mathfrak{g}) \subseteq D(G, L)$ induces on $\mathfrak{F}(\mathfrak{g})$ the structure of a nuclear Fréchet-Stein algebra (in the sense of [7], §3) which is easily seen to depend only on \mathfrak{g} . It is thus a topological algebra in the above sense. At the same time $U(\mathfrak{g})$ is a topological algebra with respect to the finest locally convex topology. We therefore have the categories $\mathcal{M}_{U(\mathfrak{g})}$ and $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ at our disposal.

Proposition 2.1. *The map*

$$U(\mathfrak{g}) \longrightarrow \mathfrak{F}(\mathfrak{g})$$

is stably flat. The topological algebra is of finite type (in the sense of [10], Def. 2.4).

As with any localization we obtain that the restriction functor θ_* identifies $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ with a full subcategory of $\mathcal{M}_{U(\mathfrak{g})}$ (cf. [9], Prop. 1.2) and leaves cohomology invariant ([loc.cit.], Prop.1.4). Since in our setting $U(\mathfrak{g})$ has the finest locally convex topology one may go one step further and pass to abstract Lie algebra cohomology.

Theorem 2.2. *Given $M, N \in \mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ the restriction functor θ_* induces natural vector space isomorphisms*

$$\mathrm{Tor}_*^{\mathfrak{F}(\mathfrak{g})}(M, N) \cong \mathrm{Tor}_*^{U(\mathfrak{g})}(M, N), \quad \mathcal{E}\mathrm{xt}_{\mathfrak{F}(\mathfrak{g})}^*(M, N) \cong \mathrm{Ext}_{U(\mathfrak{g})}^*(M, N).$$

We will prove the auxiliary proposition and the theorem in the following section.

Remarks: 1. The natural map $U(\mathfrak{g}) \rightarrow D(G, L)$ is not stably flat unless G is discrete. Indeed, if it was so would be the map $\mathfrak{F}(\mathfrak{g}) \rightarrow D(G, L)$ (cf. [5], Prop. 3.5 which remains valid in our setting). But using [2], Prop. 4.2/Cor. 4.4 this is false if G is not discrete. 2. The stable flatness of θ implies that the underlying ring homomorphism is flat. Indeed, it suffices to prove $\mathrm{Tor}_*^{U(\mathfrak{g})}(\mathfrak{F}(\mathfrak{g}), M) = 0$ for $* > 0$ and finitely generated $M \in \mathcal{M}_{\mathfrak{g}}$ and this follows, using stable flatness of θ , from choosing a resolution of M by finite free modules and equipping them with the finest locally convex topology. 3. If \mathfrak{g} is abelian the ring homomorphism underlying θ equals the inclusion of the polynomial ring into rigid analytic functions on affine space. By standard commutative algebra the ring homomorphism is then even faithfully flat.

3. THE PROOF

3.1. The hyperenveloping algebra. Let $U(\mathfrak{g})$ be the enveloping algebra of \mathfrak{g} and $\mathfrak{x}_1, \dots, \mathfrak{x}_d$ an ordered basis of \mathfrak{g} . Using the associated PBW-basis for $U(\mathfrak{g})$ we define for real $r > 1$

$$\|\sum_{\alpha} d_{\alpha} \mathfrak{X}^{\alpha}\|_{\mathfrak{x}, r} = \sup_{\alpha} |d_{\alpha}| r^{|\alpha|}$$

where $\mathfrak{X}^{\alpha} := \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d}$, $\alpha \in \mathbb{N}_0^d$. Obviously $\|\cdot\|_{\mathfrak{x}, r}$ is a vector space norm and it is easy to check that the locally convex topology induced on $U(\mathfrak{g})$ by the family of all $\|\cdot\|_{\mathfrak{x}, r}$ does not depend on the choice of basis.

Lemma 3.1. *Suppose the basis $\mathfrak{x}_1, \dots, \mathfrak{x}_d$ has integral structure constants. Then each norm $\|\cdot\|_{\mathfrak{x}, r}$ is multiplicative.*

Proof. Put $\mathfrak{X}^{\alpha} \mathfrak{X}^{\beta} =: \sum_{\gamma} c_{\alpha\beta, \gamma} \mathfrak{X}^{\gamma}$ with $c_{\alpha\beta, \gamma} \in L$. By hypothesis $\mathfrak{X}^{\alpha} \mathfrak{X}^{\beta} = \mathfrak{x}_1^{\alpha_1 + \beta_1} \cdots \mathfrak{x}_d^{\alpha_d + \beta_d} + \mathfrak{h}$ where $\|\mathfrak{h}\|_{\mathfrak{x}, r} < \|\mathfrak{x}_1^{\alpha_1 + \beta_1} \cdots \mathfrak{x}_d^{\alpha_d + \beta_d}\|_{\mathfrak{x}, r}$. Hence $\sup_{\gamma} |c_{\alpha\beta, \gamma}| r^{|\gamma|} = \|\mathfrak{X}^{\alpha} \mathfrak{X}^{\beta}\|_{\mathfrak{x}, r} = r^{|\alpha| + |\beta|}$ and therefore $|c_{\alpha\beta, \gamma}| \leq r^{|\alpha| + |\beta| - |\gamma|}$ for all α, β, γ . It follows easily from this that $\|\cdot\|_{\mathfrak{x}, r}$ is submultiplicative. Hence, it induces a filtration of $U(\mathfrak{g})$ by additive subgroups. Using the integrality of structure constants we obtain $\|\mathfrak{x}_i \mathfrak{x}_j - \mathfrak{x}_j \mathfrak{x}_i\|_{\mathfrak{x}, r} \leq r < r^2 = \|\mathfrak{x}_i \mathfrak{x}_j\|_{\mathfrak{x}, r}$. The associated graded ring is therefore a polynomial ring over $gr L$ in the principal symbols $\sigma(\mathfrak{x}_j)$. Here $gr L$ equals Laurent polynomials over the residue field of L and therefore is an integral domain. Hence, the norm must be multiplicative. \square

From [1], Prop.1.2.8 we obtain

Lemma 3.2. *The topology on $\mathfrak{F}(\mathfrak{g})$ is induced by the family of norms $\|\cdot\|_{\mathfrak{F},r}$.*

Remark: Another important power series envelope of $U(\mathfrak{g})$ is the *Arens-Michael envelope* $\hat{U}(\mathfrak{g})$ (cf. [5], §6, [9], Def. 5.1) and equals the completion of $U(\mathfrak{g})$ with respect to all submultiplicative semi-norms. By the last two lemmas we therefore have inclusions

$$U(\mathfrak{g}) \subseteq \hat{U}(\mathfrak{g}) \subseteq \mathfrak{F}(\mathfrak{g})$$

and therefore, in particular, $\hat{U}(\mathfrak{g}) \neq 0$. We leave it as an open question whether the first inclusion is stably flat (cf. [5], Thm. 6.19 for the complex case).

We assume the hypotheses of the lemma in the following and fix a basis $\mathfrak{x}_1, \dots, \mathfrak{x}_d$ with integral structure constants and abbreviate $\|\cdot\|_r := \|\cdot\|_{\mathfrak{F},r}$. Following [5] a *Hopf $\hat{\otimes}$ -algebra* is a Hopf algebra object in the braided category of topological algebras.

Lemma 3.3. *$\mathfrak{F}(\mathfrak{g})$ is a Hopf $\hat{\otimes}$ -algebra with invertible antipode.*

Proof. It is not hard to check that the usual Hopf structure on $U(\mathfrak{g})$ respects each (multiplicative) norm $\|\cdot\|_r$ and therefore extends to the completion. \square

Remark: The Hopf structure is easily seen to be dual to the one on the Hopf $\hat{\otimes}$ -algebra $C_1^{an}(G, L)$. The structure on the latter arises by functoriality in G .

3.2. Resolutions. Recall the homological standard complex $U^\bullet := U(\mathfrak{g}) \otimes_L \bigwedge^\bullet \mathfrak{g}$ with differential $\partial = \psi + \phi$ where

$$\psi(\lambda \otimes \mathfrak{x}_1 \wedge \dots \wedge \mathfrak{x}_q) = \sum_{s < t} (-1)^{s+t} \lambda \otimes [\mathfrak{x}_s, \mathfrak{x}_t] \wedge \mathfrak{x}_1 \wedge \dots \wedge \widehat{\mathfrak{x}}_s \wedge \dots \wedge \widehat{\mathfrak{x}}_t \wedge \dots \wedge \mathfrak{x}_q,$$

$$\phi(\lambda \otimes \mathfrak{x}_1 \wedge \dots \wedge \mathfrak{x}_q) = \sum_s (-1)^{s+1} \lambda \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge \dots \wedge \widehat{\mathfrak{x}}_s \wedge \dots \wedge \mathfrak{x}_q.$$

Let I_q be the collection of indices $1 \leq i_1 < \dots < i_q \leq d$ and $\lambda_q = \sum_{I \in I_q} u_I \otimes x_I \in U^q$ with $u_I \in U(\mathfrak{g})$, $x_I = \mathfrak{x}_{i_1} \wedge \dots \wedge \mathfrak{x}_{i_q} \in \bigwedge^q \mathfrak{g}$. We put for $\sum_q \lambda_q \in U^\bullet$

$$\|\sum_q \lambda_q\|_r := \sup_q r^q \sup_{I \in I_q} \|u_I\|_r$$

for $r > 1$. In this way U^\bullet becomes a normed left $U(\mathfrak{g})$ -module.

Lemma 3.4. *The differential ∂ is norm-decreasing.*

Proof. Using that for the structure constants we have $|c_{stk}| \leq 1$ and that $\|\cdot\|_r$ is multiplicative on $U(\mathfrak{g})$ we obtain

$$\begin{aligned} \|\partial(\lambda_q)\|_r &\leq \sup_{I \in I_q} \left\| \sum_{s < t} (-1)^{s+t} u_I \otimes \left(\sum_k c_{stk} \mathfrak{x}_k \right) \wedge \mathfrak{x}_1 \wedge \dots \wedge \widehat{\mathfrak{x}}_s \wedge \dots \wedge \widehat{\mathfrak{x}}_t \wedge \dots \wedge \mathfrak{x}_q \right. \\ &\quad \left. + \sum_s (-1)^{s+1} u_I \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge \dots \wedge \widehat{\mathfrak{x}}_s \wedge \dots \wedge \mathfrak{x}_q \right\|_r \\ &\leq \sup_{I \in I_q} \max \left(\sup_{s < t} r^{q-1} \|u_I\|_r, \sup_s r^q \|u_I\|_r \right) \\ &\leq \sup_{I \in I_q} r^q \|u_I\|_r = \|\lambda_q\|_r. \end{aligned}$$

\square

In the following we endow U^\bullet with the locally convex topology induced by the norms $\|\cdot\|_r$. The last result then implies that ∂ is continuous.

The augmented complex $U^\bullet \xrightarrow{\epsilon} L$ has an L -linear contracting homotopy constructed in [3], V.1.3.6.2. To review its construction we have to introduce the Koszul complex $S^\bullet := S(\mathfrak{g}) \otimes_L \bigwedge^\bullet \mathfrak{g}$ attached to the vector space \mathfrak{g} where the differential is given by the map ϕ above. Note that there is the obvious isomorphism $f : U^\bullet \xrightarrow{\cong} S^\bullet$ as L -vector spaces induced by the choice of basis $\mathfrak{x}_1, \dots, \mathfrak{x}_d$. The augmented complex $S^\bullet \rightarrow L$ comes equipped with the following contracting homotopy \bar{s} depending on the basis \mathfrak{x}_j (cf. [3], (1.3.3.4)). In case $d = 1$ it is given by the structure map $\eta : L \rightarrow S(\mathfrak{g})$ together with $\bar{s}_0 : S(\mathfrak{g}) \rightarrow S(\mathfrak{g}) \otimes \bigwedge^1 \mathfrak{g}$ defined via $\bar{s}_0(\mathfrak{x}_1^n) = \mathfrak{x}_1^{n-1} \otimes \mathfrak{x}_1$ for all $n \in \mathbb{N}$ and $\bar{s}_0(1) = 0$. In general, the definition is extended to the tensor product

$$f_1^\bullet : S_1^\bullet \otimes_L \cdots \otimes_L S_d^\bullet \xrightarrow{\cong} S^\bullet$$

by general principles (cf. [3], V.1.3.2.). Here, S_j^\bullet equals the Koszul complex of the space $L\mathfrak{x}_j$ and the latter isomorphism is induced by functoriality of S^\bullet . One obtains from this a contracting homotopy on U^\bullet as follows: pulling \bar{s} back to U^\bullet via f one obtains an L -linear map σ on U^\bullet which is precisely the one referred to in [3], V.1.3.6.2. It comes equipped with maps $\sigma_{(n)} : U^\bullet \rightarrow U^\bullet, \sigma_{(n)}^q : U^q \rightarrow U^{q+1}$ (cf. [loc.cit.], Lem. V.1.3.5) having the property: for fixed $x \in U^q$ the sequence $\sigma_{(n)}(x)$ becomes stationary ([loc.cit.], remark after formula V.1.3.6.2). Hence,

$$s(x) := \lim_n \sigma_{(n)}(x)$$

defines the desired map s .

Lemma 3.5. *The map s is continuous on the topologized complex U^\bullet .*

Proof. First of all, we may endow the augmented complex $S^\bullet \rightarrow L$ with a family of norms $\|\cdot\|_r, r > 1$ such that the isomorphism f becomes isometric. We now prove the claim in two steps. 1. Fixing a norm $\|\cdot\|_r$ on S^\bullet the homotopy \bar{s} is norm-decreasing:

By induction on $\dim_L \mathfrak{g}$ we may endow the left-hand side of the isomorphism f_1^\bullet with the following norm:

$$\|\lambda\|_r := \sup_{s+t=q} \inf_{(\lambda_s), (\mu_t)} \|\lambda_s\| \|\mu_t\|_r$$

where $\lambda \in (S_i^\bullet \otimes_L S_j^\bullet)^q = \oplus_{s+t=q} S_i^s \otimes_L S_j^t$ is of the form $\lambda = \sum_{s+t=q} (\sum \lambda_s \otimes \mu_t)$ and the infimum is taken over all possible representations of the (s, t) -component of λ . We claim that f_1^\bullet is isometric. Again by induction we are reduced to prove the claim for f_2^q where

$$f_2^\bullet : S_{<d}^\bullet \otimes_L S_d^\bullet \xrightarrow{\cong} S^\bullet$$

and $S_{<d}^\bullet$ equals the Koszul complex of $\oplus_{j<d} L\mathfrak{x}_j$. Fix $q \geq 0$. By definition of $\|\cdot\|_r$ the decomposition $(S_{<d}^\bullet \otimes_L S_d^\bullet)^q = \oplus_{s+t=q} S_{<d}^s \otimes_L S_d^t$ is orthogonal. By definition of f_2^q and since the elements $\{1 \otimes x_{I_q}\}$ are orthogonal in S^q , f_2^q preserves this orthogonality in S^q . It therefore suffices to fix $s+t = q$ and prove $\|f_2(\lambda)\|_r = \|\lambda\|_r$ for $\lambda \in S_{<d}^s \otimes_L S_d^t$. In both cases ($s = q$ and $s = q - 1$) this is a straightforward computation whence f_1^\bullet is indeed isometric. Next we prove that \bar{s} is norm-decreasing on the left-hand side of the isomorphism f_1^\bullet . For $d = 1$ this follows since η and \bar{s}_0 are certainly norm-decreasing. By induction we may suppose that this is true on the complex $S_{<d}^\bullet$ and consider the tensor product $S_{<d}^\bullet \otimes_L S_d^\bullet$. Let $\lambda \in (S_{<d}^\bullet \otimes_L S_d^\bullet)^q$. Suppose

$q = 0$ and hence $\lambda \in L$. It is then clear that $\|s(\lambda)\|_r = \|\eta(\lambda) \otimes 1\|_r = \|\lambda\|_r$ where the first identity follows from formula [loc.cit.], V.1.3.2.2. So assume $q > 0$. Write $\lambda = \sum_{s+t=q} (\sum \lambda_s \otimes \mu_t)$. Then

$$\bar{s}(\lambda) = \sum_{s+t=q, s>0} \sum \bar{s}(\lambda_s) \otimes \mu_t + \sum_{s+t=q, s=0} \sum \bar{s}(\lambda_s) \otimes \mu_t + \eta\epsilon(\lambda_s) \otimes \bar{s}(\mu_t).$$

according to the formulas [loc.cit.], V.1.3.2.2/1.3.2.3. Using the induction hypothesis on the right-hand side one obtains $\|\bar{s}(\lambda)\|_r \leq \|\lambda\|_r$ as desired.

2. The lemma follows: Fix a norm $\|\cdot\|_r$ on U^\bullet . By the first step the L -linear map $\sigma := f^{-1} \circ \bar{s} \circ f$ on U^\bullet is norm-decreasing. The augmentation $\epsilon : U^0 \rightarrow L$ and the differential ∂ are also norm-decreasing (the latter by Lem. 3.4). Invoking the maps $\sigma_{(n)}$ from above we deduce from $\sigma_{(0)} = \sigma$ and the formula

$$\sigma_{(n)} - \sigma_{(n-1)} = \sigma(1 - \epsilon - \partial\epsilon - \epsilon\partial)^n$$

([loc.cit.], V.1.3.5.4) by induction that all $\sigma_{(n)}$ are norm-decreasing. Now the contracting homotopy s of U^\bullet is defined as the pointwise limit $s^q(x) := \lim_n \sigma_{(n)}^q(x)$, $x \in U^q$. It is thus norm-decreasing since the sequence $\sigma_{(n)}^q(x)$ for $n \rightarrow \infty$ becomes eventually stationary. Since $r > 1$ was arbitrary the homotopy s is therefore continuous with respect to the locally convex topology induced by the family $\|\cdot\|_r$. \square

3.3. Stable flatness. We prove the proposition and the theorem of section 2.

Proof. It is easy to check that [5], Prop. 3.7 remains valid when \mathbb{C} is replaced by L and the complete projective tensor product by the complete inductive tensor product. Thus, by this result and Lem. 3.3 it suffices to see that the acyclic complex $\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} U^\bullet = \mathfrak{F}(\mathfrak{g}) \otimes_L \dot{\bigwedge} \mathfrak{g}$ in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ has a continuous contracting homotopy. Since this complex equals the Hausdorff completion of the topologized complex U^\bullet such a homotopy is obtained from extending s by continuity (cf. Lem. 3.5) to the completion. Hence we have stable flatness. Moreover, $\mathfrak{F}(\mathfrak{g})$ has now a finite projective resolution in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$ by free (left) $\mathfrak{F}(\mathfrak{g})$ -modules. The adjoint action of \mathfrak{g} on $U(\mathfrak{g})$ turns the homological standard resolution of $U(\mathfrak{g})$ into a bimodule resolution in the usual way. Using integrality of structure constants it is easy to see that this action extends to $\mathfrak{F}(\mathfrak{g})$ whence the base extended resolution is therefore a finite bimodule resolution of $\mathfrak{F}(\mathfrak{g})$. Thus, $\mathfrak{F}(\mathfrak{g})$ is of finite type. \square

Proof. Let $M, N \in \mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$. Any projective resolution $P_\bullet \rightarrow M$ in $\text{Mod}(\mathfrak{g})$ by abstract free $U(\mathfrak{g})$ -modules is a projective resolution in $\mathcal{M}_{U(\mathfrak{g})}$ when endowed with the finest locally convex topology. It is easy to check that [5], Prop. 3.3 remains valid in our setting whence the natural map $\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} M \rightarrow M$ is an isomorphism in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$. Hence, by stable flatness $\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} P_\bullet \rightarrow M$ is a projective resolution of M in $\mathcal{M}_{\mathfrak{F}(\mathfrak{g})}$. The claims follow now from the isomorphisms of complexes

$$\mathcal{L}_{\mathfrak{F}(\mathfrak{g})}(\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} P_\bullet, N) \simeq \mathcal{L}_{U(\mathfrak{g})}(P_\bullet, N) \simeq \text{Hom}_{\mathfrak{g}}(P_\bullet, N)$$

and

$$N \hat{\otimes}_{\mathfrak{F}(\mathfrak{g})} (\mathfrak{F}(\mathfrak{g}) \hat{\otimes}_{U(\mathfrak{g})} P_\bullet) \simeq N \hat{\otimes}_{U(\mathfrak{g})} P_\bullet \simeq N \otimes_{U(\mathfrak{g})} P_\bullet$$

where the last isomorphisms in both rows follow from the fact that P_\bullet carries the finest locally convex topology in each degree. \square

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